Matrix Algebra and Is Applications

Problem Set Before Quiz #2

1. If **V** is **W** are orthogonal subspaces, show that the only vector they have in common is the zero vector, i.e., $\mathbf{V} \cap \mathbf{W} = \{0\}$.
2. Find a vector x orthogonal to the row space of A, and a vector y orthogonal to the column space, and a vector z orthogonal to the nulspace:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

3. Draw Figure 3.4 to show each subspace for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

4. Find the matrix that projects every point in R^3 onto the line, which is the intersection of the planes $x_1+x_2+x_3=0$ and $x_1-x_3=0$.
5. Suppose the values $b_1=1$ and $b_2=7$ at times $t_1=1$ and $t_2=2$ are fitted by a line $b=Dt$ through the origin. Find $\hat{D}$ by least square and sketch the observations with the best-fit line. Find $\hat{D}$ by projection and sketch the projection of b onto the column space of t.
6. If V is the subspace spanned by (1, 1, 0, 1) and (0, 0, 1, 0), find
   (a) a basis for the orthogonal complement $V^{\perp}$
   (b) the projection matrix P onto V
   (c) the vector in V closest to the vector b = (0, 1, 0, -1) in $V^{\perp}$
7. If P is the projection matrix onto a k-dimensional subspace S of the whole space $\mathbf{R}^n$, what is the column space of P and what is its rank?
8. If u is a unit vector, show that $Q=I-2uu^T$ is an orthogonal matrix and is a reflection transformation. Compute Q when u^T=(1/2, 1/2, 1-/2, -1/2) and explain what Q does to x with Qx.
9. Suppose the values $b_1=1$ and $b_2=7$ at times $t_1=1$ and $t_2=2$ are fitted by a line $b=Dt$ through the origin. Find $\hat{D}$ by least square and sketch the observations with the best-fit line. Find $\hat{D}$ by projection and draw the projection of b onto the column space of t.
10. Project $b=(0, 3, 0)^T$ onto each of the orthonormal vectors $a_1=(2/3, 2/3, -1/3)^T$ and a_2=(-1/3, 2/3, 2/3). Then, find its projection p onto the plane spanned by a_1 and a_2. Also find the projection onto a_3=(2/3, -1/3, 2/3) and add the three projections. Why is $P=a_1(a_1)^T+a_2(a_2)^T+a_3(a_3)^T$ equal to I?
11. Show that $Q=I-2uu^T$ is an orthogonal matrix.
12. (a) Find the bases for the null space and the row space of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

(b) Split $x = (3, 3, 3)^T$ into a row-space component x_r and a null-space component x_n. (c) Find the pseudoinverse of A

(d) Let $Ax=(9, 21)^T$. Recover the row space component of x.

(e) Show that the pseudoinverse found in (c) is the right inverse of A.

13. Project the vector $b=(1, 2)$ onto a 2-dimensional basis with two vectors, (1, 0) and (1, 1), and show that, unlike the orthogonal basis, the sum of the two projections does not equal to b.

14. Show that an orthogonal matrix that is upper triangular must be diagonal.

15. Factor

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

into QR and find the least squares solution of $Ax=b$ if $b=(1, 2, 7)^T$?

16. Find the best parabola: $y = C + Dt + Et^2$ fit to the following measurements:

$y = 2$ at $t = -1$,

$y = 0$ at $t = 0$,

$y = -3$ at $t = 1$,

$y = -5$ at $t = 2$.

Find your approximate solution by QR factorization and plot the observations together with the best-fit parabola using Excel.

17. Find the Fourier coefficients a_0, a_1, b_1, a_2, b_2 of the step function $y(x)$ which equals 1 on the interval $0 \le x \le \pi/2$ and -1 on the remaining interval $\pi/2 < x < 2\pi$. Plot $y(x)$ and the Fourier series on the same figure (you may use Excel to create the figure).

18. Find the determinants of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

by eliminations and possible row exchanges.

19. What is the volume of the parallelepiped with the four of its vertices at (1, 1, 1), (-1, 2, 2), (2, -1, 2) and (2, 2, -1)?

20. If a 6 by 6 determinant is expanded into the sum of 6^6 determinants by splitting each row into 6 coordinate directions, how many determinant terms in the expansion are sure to be zero if $a_{21}=0$?

21. Find the determinant of A, if $a_{ij}=i+j$.