Matrix Algebra and Is Applications

Problem Set for Chapters 5 and 6

1. Show that the determinant equals the product of the eigenvalue by first factoring the characteristic polynomial into det(***A***- λ***I***)=(λ_1- λ)(λ_2- λ)…(λn- λ) and then making a clever choice of λ.

2. Show that the eigenvalues of ***A*** equal the eigenvalues of $\boldsymbol{A^T}$. Show by an example that the eigenvectors of ***A*** and $\boldsymbol{A^T}$ are not the same.

3. Suppose the matrix ***A*** has eigenvalues 0,1,2 with eigenvectors v_0, v_1, v_2. Describe the nullspace and the column space of ***A***. Solve the equation $Ax = v_1 + v_2$. Show that $Ax = v_0$ has no solution.

4. Which of these matrices cannot be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

5. (a) If $A^2 = I$, what are the possible eigenvalues of ***A***?
   (b) If this ***A*** is 2 by 2, and not ***I*** or –***I***, find its trace and determinant?
   (c) If the first row is (3, -1), what is the second row?

6. Let ***P*** be the projection matrix that projects any vector in R^4 onto $x_1+x_2+x_3+x_4=0$. Find the eigenvalues and eigenvectors of ***P***.

7. If each number is the average of the two previous numbers, $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$, set up the matrix ***A*** and diagonalize it.
   (a) Find a formula for G_k.
   (b) Compute G_k as $k \to \infty$.
   (c) Starting from $G_0 = 0$ and $G_1 = \frac{1}{2}$, compute G_∞.

8. What are the limits as $k \to \infty$ (the steady states) of

$$\begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}^k ?$$

9. Write out the matrix A^H and compute $C = A^H A$ if

$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}$$

What is the relation between C and C^H? Does it hold whenever C is constructed from some $A^H A$?

10. Rewrite the following matrices in the form $\lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

11. Write one significant fact about the eigenvalues of each of the following
    (a) A real symmetric matrix
    (b) A stable matrix (solutions of $du/dt=Au$ approach zero)
    (c) An orthogonal matrix
    (d) A Markov matrix
    (e) A defective (nondiagonalizable) matrix
    (f) A singular matrix

12. Diagonalize the following 2 by 2 skew-Hermitian matrix

$$K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}.$$

Compute $e^{Kt}=Se^{\Lambda t}S^{-1}$, and verify that e^{Kt} is unitary.

13. Every matrix Z can be split into a Hermitian and a skew-Hermitian part, i.e., $Z=A+K$ where $A=(Z+Z^H)/2$. Find the formula for K and split the following matrices into $A+K$.

$$Z = \begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix}, \text{ and } \quad Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$$

14. Diagonalize the following unitary matrix V to reach $V=U\Lambda U^H$. (Remember all $|\lambda|=1$)

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

15. (a) What matrix M changes the basis $V_1=(1, 1)$, $V_2=(1, 4)$ to the basis $v_1=(2, 5)$, $v_2=(1, 4)$? (Hint: the columns of M come from expressing V_1 and V_2 as combinations $\Sigma m_{ij} v_i$ of the v's.)
    (b) For the same two bases, express the vector (3, 9) as a combination $c_1V_1+c_2V_2$ and also as $d_1v_1+d_2v_2$. Check numerically that M connects c to d: $Mc=d$.

16. Let $A=\begin{bmatrix}5 & 4\\4 & 5\end{bmatrix}$ and the quadratic function be $\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}5 & 4\\4 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

(i) Factor A into LDU and expand the quadratic function into a summation of two quadratic terms.

(ii) Diagonalize A into $Q\Lambda Q^T$ and expand the quadratic function into a summation of two quadratic terms.

(iii) Compare results of (i) and (ii) and use them to determine whether the quadratic function is definite, semi-definite or indefinite.

17. Give a quick reason why each of these statements is true:

(i) Every positive definite matrix is invertible.

(ii) The only positive definite projection matrix is $P=I$

(iii) A diagonal matrix with positive diagonal entries is positive definite.

(iv) A symmetric matrix with a positive determinant might not be positive definite.

18. Find the singular decomposition of

$$A=\begin{bmatrix}2 & 2\\1 & 0\end{bmatrix}$$

19. Compute the polar decomposition $A=QS$.

$$A=\frac{1}{\sqrt{10}}\begin{bmatrix}10 & 6\\0 & 8\end{bmatrix}$$

Knowing Q, find the reverse form $A=S'Q$.

20. Find the singular value decomposition and the psuedoinverse of

$$A=\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \quad B=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\end{bmatrix} \text{ and } C=\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

21. If B is similar to A and C is similar to B, show that C is similar to A. Which matrices are similar to I.